Quasisymmetric sewing in Rigged Teichmüller space

David Radnell

Dept. of Mathematics,
University of Michigan,
USA

and

Max Planck Institute for
Mathematics, Bonn,
Germany

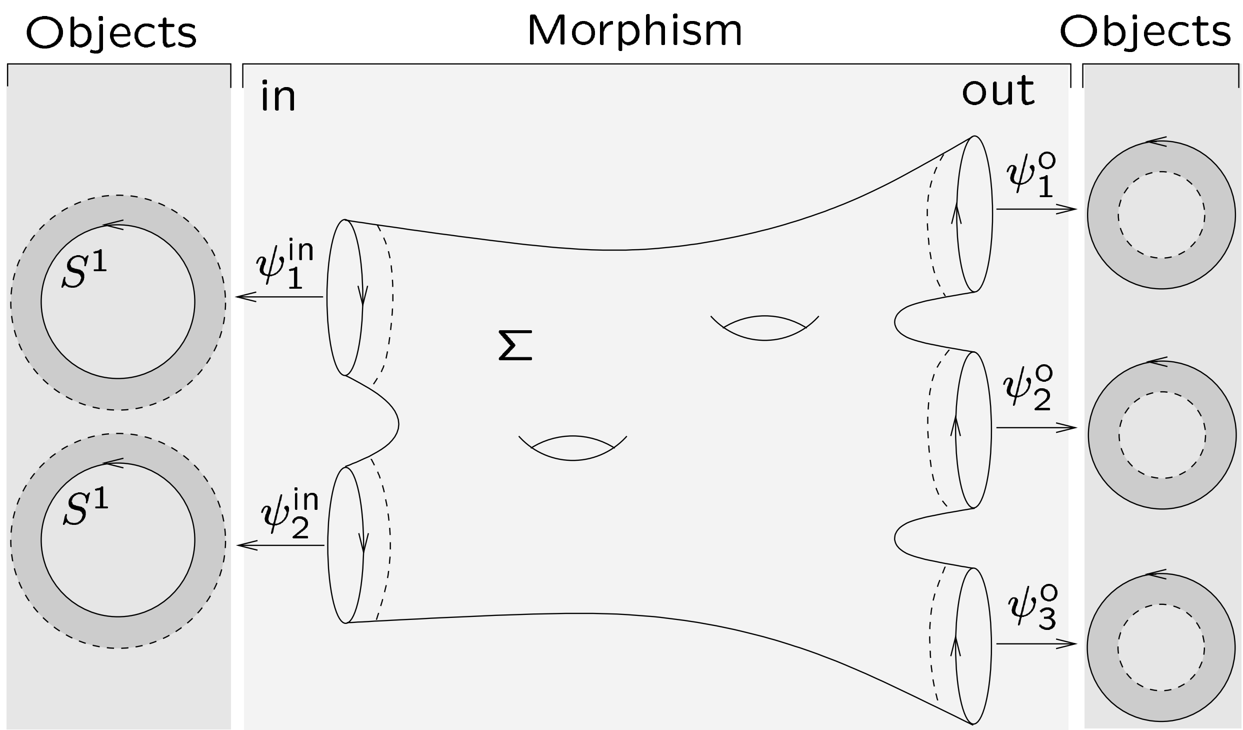
Eric Schippers

Dept. of Mathematics,
University of Manitoba,
Canada

Motivation and Applications

Definition/construction of conformal field theory (CFT).
(G. Segal '87)

Formalizes: $\text{Amplitude} = \int_{\{f:\Sigma\to M\}} e^{-S_\Sigma(f)}\mathcal{D}f$



Projective Functor

$$H \otimes H \xrightarrow{\quad A([\Sigma,\psi]) \quad} H \otimes H \otimes H$$

Weakly (chiral) CFTs:
$A([\Sigma,\psi])$ depends holomorphically on $[\Sigma,\psi]$.

- Deeper mathematical structure.
- Construct using Vertex Operator Algebras ($g = 0, 1$ complete – Huang, Zhu).

Basic Problem

For a rigorous definition of weakly CFT we need to make sense of

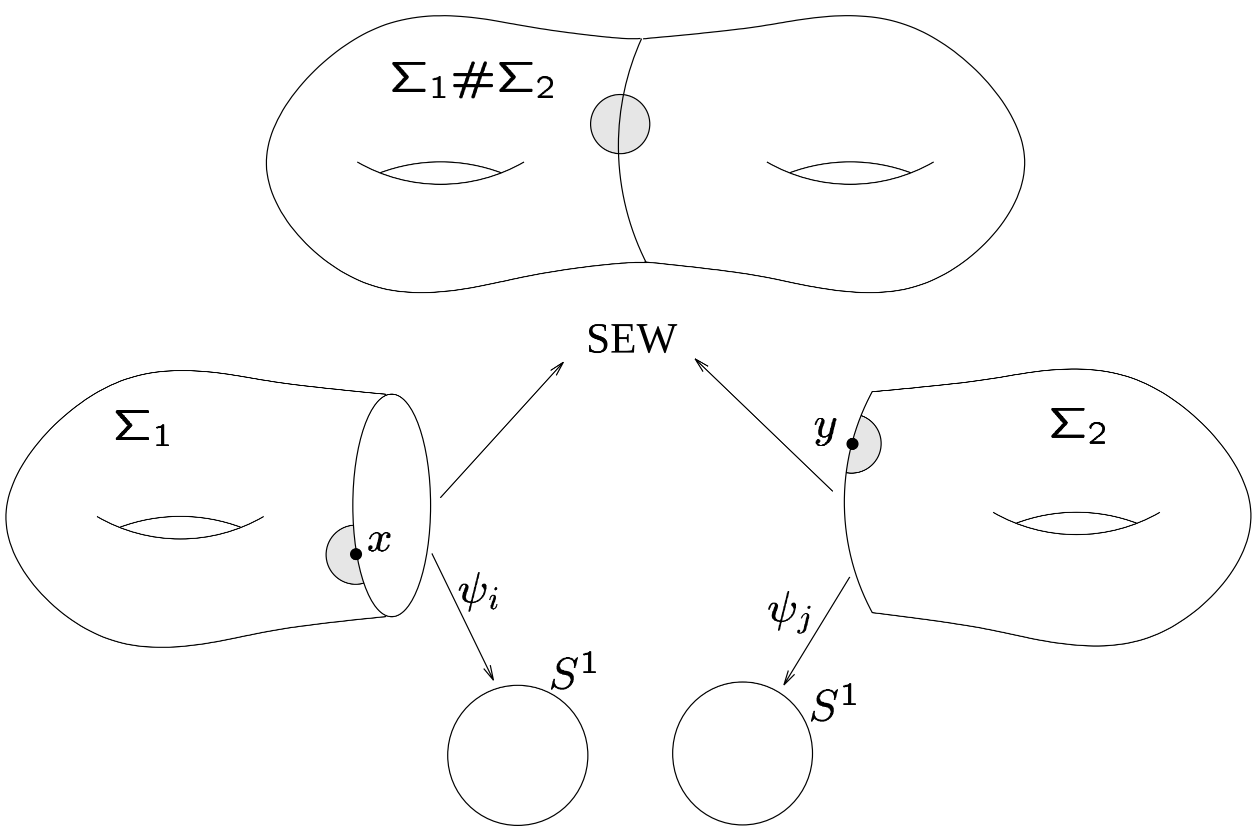
"$A([\Sigma, \boldsymbol{\psi}])$ depends holomorphically on $[\Sigma, \boldsymbol{\psi}]$".

Theorem.

1. The moduli space of rigged surfaces is a complex manifold.
2. The sewing operation is a holomorphic map between rigged moduli spaces.

Proof. $\boldsymbol{\psi}$ analytic: Huang 91 ($g = 0$), R. 03 (all g).

Today
$\boldsymbol{\psi}$ quasisymmetric: RS 05



$$\Sigma_1\#\Sigma_2 = (\Sigma_1 \sqcup \Sigma_2)/\big(\psi_i(x) = \psi_j(y)\big)$$

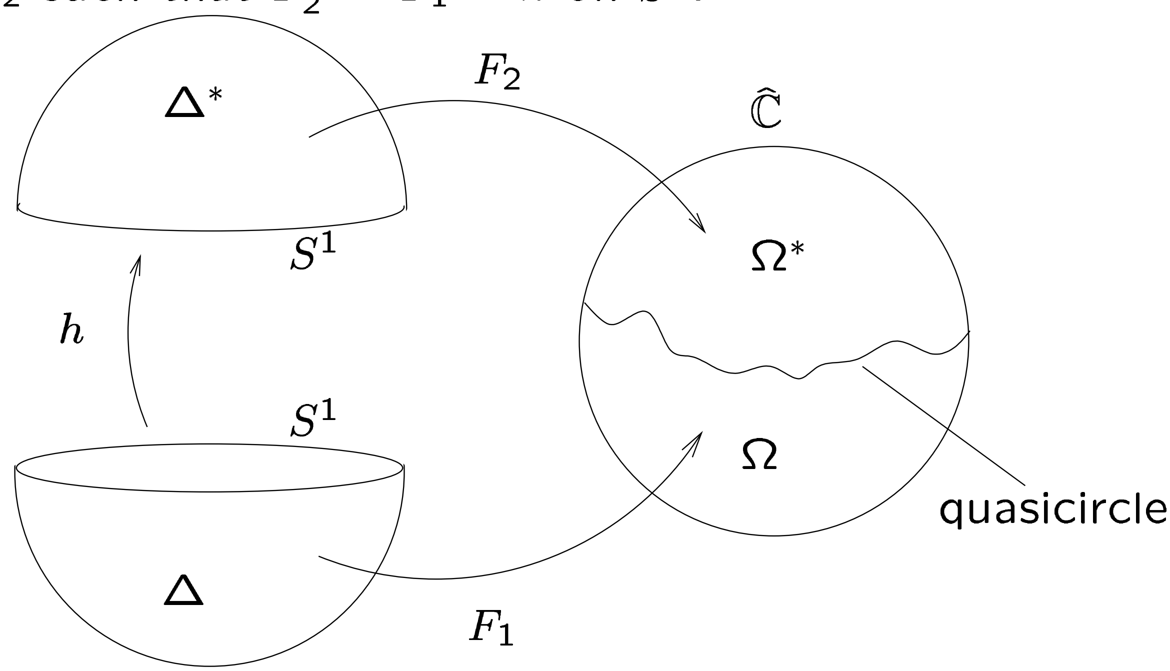
Conformal Welding

Δ – unit disk, $\quad \Delta^* = \hat{\mathbb{C}} \setminus \bar{\Delta}$

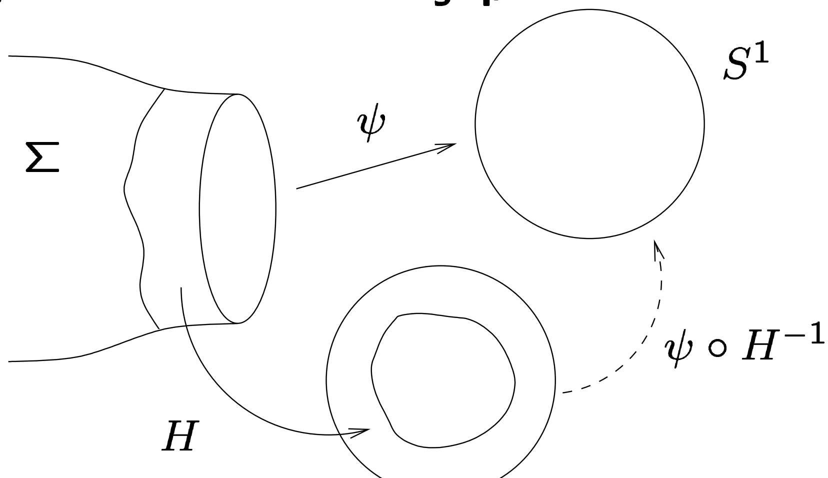
$w : \mathbb{C} \to \mathbb{C}$ is **quasiconformal** $\iff$ its circular dilation is bounded.

$h : S^1 \to S^1$ is **quasisymmetric** $\iff$ there exists a quasiconformal extension $w : \Delta \to \Delta$.

Conformal welding: There exists conformal F_1 and F_2 such that $F_2^{-1} \circ F_1 = h$ on S^1.



Quasisymmetric boundary parametrization:



Quasisymmetric Sewing

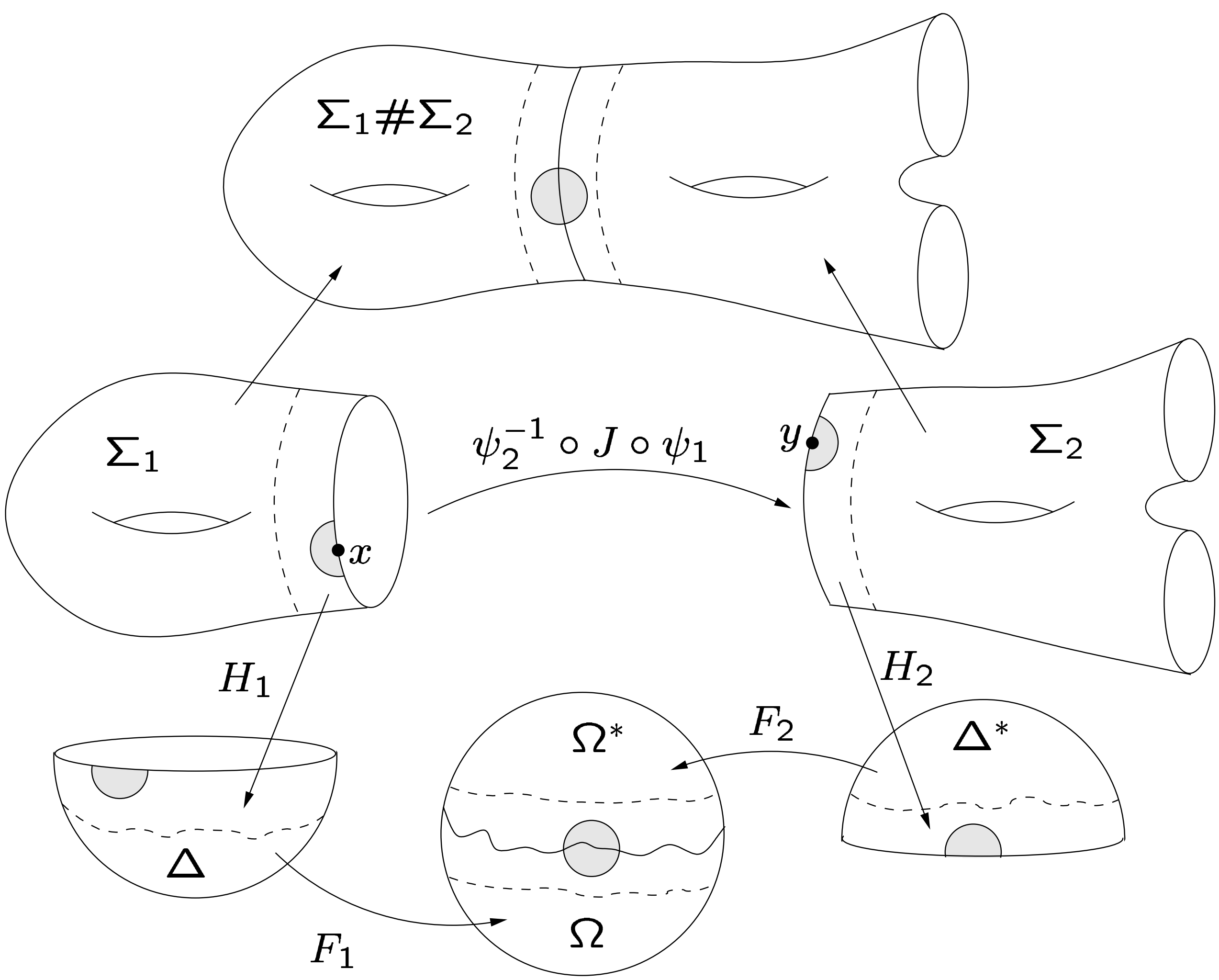
Quasisymmetrically rigged surfaces: (Σ_1, ψ_1), (Σ_2, ψ_2)

$$\Sigma_1 \# \Sigma_2 = (\Sigma_1 \sqcup \Sigma_2)/\sim$$
$$x \sim y \Longleftrightarrow (\psi_2^{-1} \circ J \circ \psi_1)(x) = y$$

$\psi_i : \partial\Sigma_i \to S^1$

$J(z) = 1/z$

Define charts using:



Proposition. This gives the unique complex structure on $\Sigma_1 \# \Sigma_2$ which is compatible with Σ_1 and Σ_2.

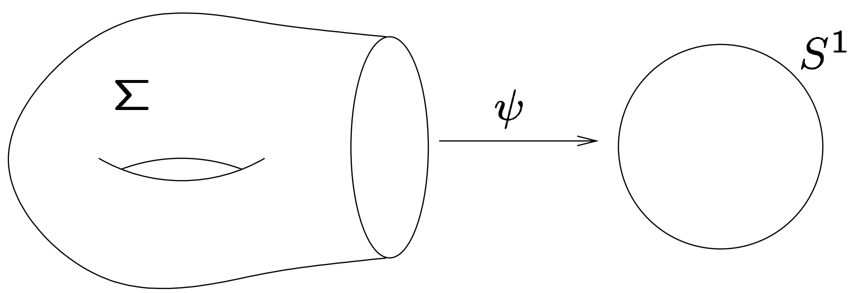
Rigged Moduli Spaces

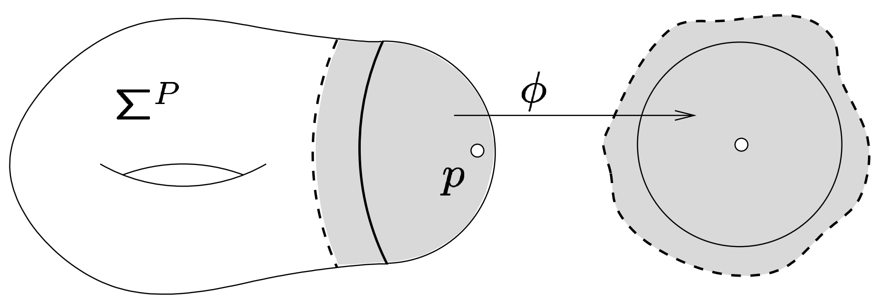
Fix: n – number of boundary components, g – genus.
Notation: $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)$ – list of parametrizations.

Border Model: $\widetilde{\mathcal{M}}^B(g,n) = \{(\Sigma, \boldsymbol{\psi})\}/\sim$

$(\Sigma_1, \boldsymbol{\psi}) \sim (\Sigma_2, \boldsymbol{\psi}') \Longleftrightarrow \exists\, \sigma : \Sigma_1 \to \Sigma_2$ and $\psi_i' \circ \sigma = \psi_i$

Sew on puncutred disks: $\Sigma^P = \Sigma \# (\overline{\Delta}_0)^n$.





Puncture model:
Σ^P – surface with punctures $\mathsf{p} = (p_1, \ldots, p_n)$.

$$\mathcal{O}_{\mathsf{qc}}(p_i) = \{\phi : \mathrm{nbh}(p_i) \to \mathbb{C} \,|\, \Delta \subset \mathrm{im}(\phi),\ \phi \text{ is quasiconformal},\ \phi^{-1} \text{ is biholomorphic on } \Delta\}$$

$\widetilde{\mathcal{M}}^P(g,n) = \{\Sigma^P, \boldsymbol{\phi}\}/\sim$

Proposition. $\widetilde{\mathcal{M}}^B(g,n) \simeq \widetilde{\mathcal{M}}^P(g,n)$

Rigged Teichmüller Spaces

Fix: Rigged base surface $(\Sigma, \boldsymbol{\tau})$ and $\Sigma^P = \Sigma \#_\tau (\overline{\Delta}_0)^n$.

Given Σ_1 and quasiconformal $f : \Sigma \to \Sigma_1$, write (Σ, f, Σ_1).

Teichmüller space: $T(\Sigma) = \{(\Sigma, f, \Sigma_1)\}/\sim$.

$(\Sigma, f, \Sigma_1) \sim (\Sigma, g, \Sigma_2) \iff \exists\, \sigma : \Sigma_1 \to \Sigma_2$ such that $g^{-1} \circ \sigma \circ f \approx \mathrm{id}$ (rel. boundary)

Rigged Teichmüller spaces

Border Model: $\widetilde{T}^B_\#(\Sigma) = \{(\Sigma, f, \Sigma_1, \boldsymbol{\psi}_1)\}/\sim_B$

For $\sim_B$, homotopy not necessarily "rel. boundary" and $\boldsymbol{\psi}_2 \circ \sigma = \boldsymbol{\psi}_1$.

Puncture Model: $\widetilde{T}^P(\Sigma^P) = \{(\Sigma^P, f, \Sigma^P_1, \boldsymbol{\phi}_1)\}/\sim_P$

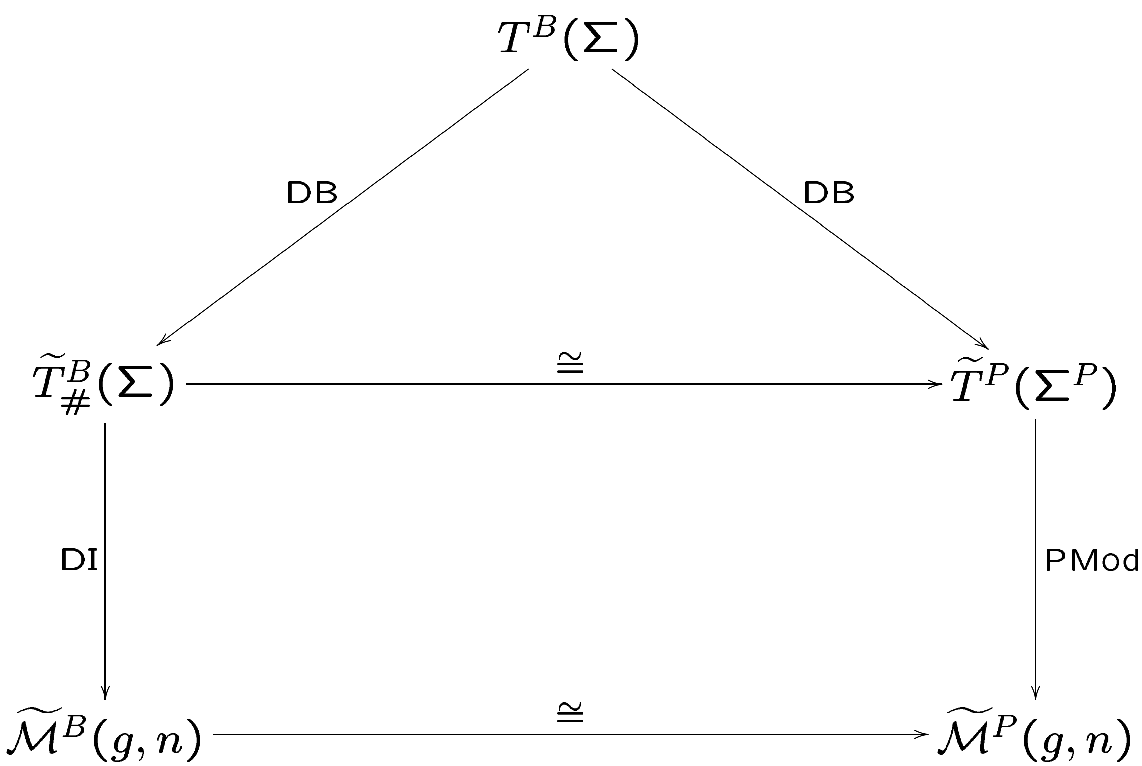
$\boldsymbol{\phi}_1 \in \mathcal{O}_{\mathrm{qc}}(\boldsymbol{p})$. For $\sim_P$, $\boldsymbol{\phi}_2 \circ \sigma = \boldsymbol{\phi}_1$.

Theorem. $\widetilde{T}^B_\#(\Sigma) \cong \widetilde{T}^P(\Sigma^P)$

Proof. Sew on caps. Need results on extensions of quasiconformal maps. Use the extended λ-lemma (Mañé-Sad-Sullivan '83, Słodkowski '91).

Complex Structures

$$
\begin{aligned}
\mathrm{QC}_0(\Sigma) &= \{\rho : \Sigma \to \Sigma \mid \rho \approx \mathrm{id}\ \text{(rel. boundary)} \\
\mathrm{PMod}(\Sigma) &= \mathrm{QC}(\Sigma)/\mathrm{QC}_0(\Sigma) \\
\mathrm{PMod\,I}(\Sigma) &= \{[\rho] \in \mathrm{PMod}(\Sigma) \mid \text{on } \partial\Sigma, \rho = \mathrm{id}\} \\
\mathrm{DB}(\Sigma) &= \text{the subgroup } \big\langle \text{“boundary” Dehn twists} \big\rangle \\
\mathrm{DI}(\Sigma) &= \text{the subgroup } \big\langle \text{“internal” Dehn twists} \big\rangle
\end{aligned}
$$



Main Theorem. All the spaces are complex Banach manifolds with structure inherited from $T^B(\Sigma^B)$.

Example:

$$
\begin{aligned}
T^B(\Sigma)/\mathrm{PMod\,I}(\Sigma) &\xrightarrow{\cong} \widetilde{\mathcal{M}}^B(g,n) \\
[\Sigma, f, \Sigma_1] &\longmapsto [\Sigma_1, \boldsymbol{\tau} \circ f^{-1}]
\end{aligned}
$$

$\boldsymbol{\tau} \circ f^{-1} : \partial\Sigma_1 \to S^1$ is quasisymmetric.

Sewing

Fix rigged base spaces $(X, \boldsymbol{\tau})$ and $(Y, \boldsymbol{\eta})$.

Sew the ith boundary of X to the jth boundary of Y.

Teichmüller space level:

$\big([X, f, X_1], [Y, g, Y_1]\big) \longmapsto [X\#Y, f \cup g, X_1\#Y_1]$
X_1 and Y_1 are sewn using $\tau_i \circ f^{-1}$ and $\eta_j \circ g^{-1}$.

Moduli space level:

$\big([X_1, \boldsymbol{\psi}], [Y_1, \boldsymbol{\psi}'\,]\big) \longmapsto (X\#Y, \widehat{\boldsymbol{\psi}\boldsymbol{\psi}'})$
Sew using ψ_i and ψ'_j.

$$\begin{array}{ccc}
L^\infty_{(-1,1)}(X)_1 \times L^\infty_{(-1,1)}(Y)_1 & \xrightarrow{\mathcal{S}} & L^\infty_{(-1,1)}(X\#Y)_1 \\
\downarrow & & \downarrow \\
T^B(X) \times T^B(Y) & \xrightarrow{\mathcal{S}T} & T^B(X\#Y) \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{M}}^B(X) \times \widetilde{\mathcal{M}}^B(Y) & \xrightarrow{\mathcal{S}\mathcal{M}} & \widetilde{\mathcal{M}}^B(X\#Y)
\end{array}$$

$$S(\mu, \nu) = \mu \cup \nu = \begin{cases} \mu & \text{on } X \\ \nu & \text{on } Y \end{cases}$$

The diagram commutes.

Theorem. The sewing maps are holomorphic.

Summary

In conformal field theory the basic geometric objects are rigged Riemann surfaces $(\Sigma, \boldsymbol{\psi})$ where ψ_i are (analytic) parametrizations.

Results. We generalized to the case where $\psi_i : \partial_i\Sigma \to S^1$ are quasisymmetric and obtained the following:

1. The rigged Moduli spaces are complex manifolds.
2. The complex structure is inherited from $T^B(\Sigma)$.
3. The sewing operation is holomorphic.
4. The puncture and border models are equivalent.

These are needed in the definition of chiral CFT.

Application/benefits of the quasisymmetric case:

- Direct description of the border model.
- Application to the construction of CFT from vertex operator algebras.
- Connections to geometric function theory (to be explored).
- For all (g, n), $\widetilde{M}^B(g, n)$ sits inside $T(1) = T(\Delta)$. Adds weight to the old idea of using $T(1)$ as the space of all paths in string theory.