From infinite-dimensional Teichmüller theory to conformal field theory and back.

David Radnell

Department of Mathematics and Statistics
American University of Sharjah
Sharjah, UAE

MPIM, June 21, 2010

Congratulations to Alan Carey on his 60th birthday.

Table of contents

Introduction

Conformal Field Theory (CFT):

- Special class of 2D quantum field theories.
- First mathematical definition (G. Segal, Kontevich $\approx$ 1986)
- Deeply connected to algebra, topology and analysis.

Introduction

Conformal Field Theory (CFT):

- Special class of 2D quantum field theories.
- First mathematical definition (G. Segal, Kontevich $\approx$ 1986)
- Deeply connected to algebra, topology and analysis.

Complex analysis/geometry:

- (∞-dim) Teichmüller space of Riemann surfaces
- Sewing=gluing=welding
- Quasiconformal mappings

Introduction

Conformal Field Theory (CFT):

- Special class of 2D quantum field theories.
- First mathematical definition (G. Segal, Kontevich $\approx$ 1986)
- Deeply connected to algebra, topology and analysis.

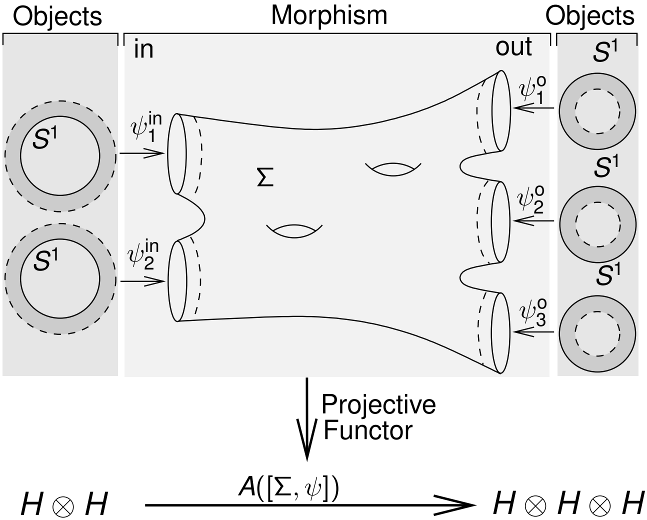
Complex analysis/geometry:

- (∞-dim) Teichmüller space of Riemann surfaces
- Sewing=gluing=welding
- Quasiconformal mappings

Our General Aim:

- Provide a natural **analytic** setting for the rigorous definition of CFT in higher genus. Definitions and Theorems.
- Use CFT ideas to prove new results in Teichmüller theory and geometric function theory.

Motivation/Application: Conformal Field Theory



The construction of CFT using Vertex Operator Algebras is nearing completion.

In the Conformal Field Theory (CFT) definition.

- Basic object: Rigged moduli space =
  { Riemann surf. with parametrized boundary } / conf. equiv.
- Basic operation: Sewing surfaces.
- The sewing operation must be a holomorphic operation on the rigged moduli spaces.

In the Conformal Field Theory (CFT) definition.

- Basic object: Rigged moduli space = { Riemann surf. with parametrized boundary } / conf. equiv.
- Basic operation: Sewing surfaces.
- The sewing operation must be a holomorphic operation on the rigged moduli spaces.

Teichmüller theory

- Teichmüller space = complex Banach manifold.
- Moduli space = Teichmüller space / mapping class group

In the Conformal Field Theory (CFT) definition.

- Basic object: Rigged moduli space =
  { Riemann surf. with parametrized boundary } / conf. equiv.
- Basic operation: Sewing surfaces.
- The sewing operation must be a holomorphic operation on the rigged moduli spaces.

Teichmüller theory

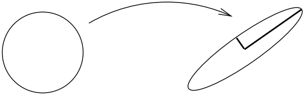
- Teichmüller space = complex Banach manifold.
- Moduli space = Teichmüller space / mapping class group

Some of our results (2006 - 2009): Teichmüller theory $\Longleftrightarrow$ CFT

- $\Longrightarrow$ The rigged moduli space can be obtained (as a complex Banach manifold) from the usual Teichmüller space.
- $\Longrightarrow$ The sewing operation is holomorphic.
- $\Longleftarrow$ A fiber structure and new coordinates for Teichmüller space.

Quasiconformal Maps

$f : \Omega \subset \mathbb{C} \to \mathbb{C}$. Homeomorphism. L^1_{loc} derivatives.

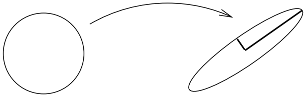
Jacobian(f) : 

Circular Dilatation = major axis / minor axis.

Quasiconformal Maps

$f : \Omega \subset \mathbb{C} \to \mathbb{C}$. Homeomorphism. L^1_{loc} derivatives.

Jacobian(f) :



Circular Dilatation = major axis / minor axis.

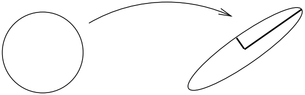
Definition

f is K**-quasiconformal** if its circular dilatation is globally bounded by K.

Quasiconformal Maps

$f : \Omega \subset \mathbb{C} \to \mathbb{C}$. Homeomorphism. L^1_{loc} derivatives.

Jacobian(f) :



Circular Dilatation = major axis / minor axis.

Definition

f is K**-quasiconformal** if its circular dilatation is globally bounded by K.

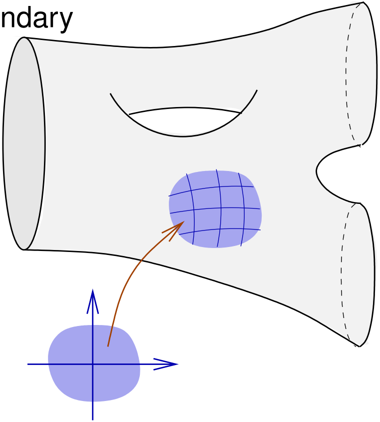
Definition

$h : S^1 \to S^1$ is **quasisymmetric** if and only if it can be extended to a quasiconformal mapping of the disk to itself.

Note: There are numerous equivalent definitions.

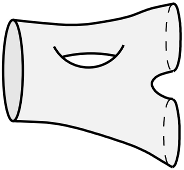
Basic Objects

- Riemann Surfaces with boundary



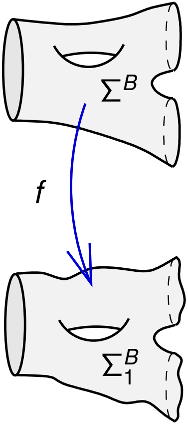
Basic Objects

- Riemann Surfaces with boundary
- Fix the genus and number of boundary components.



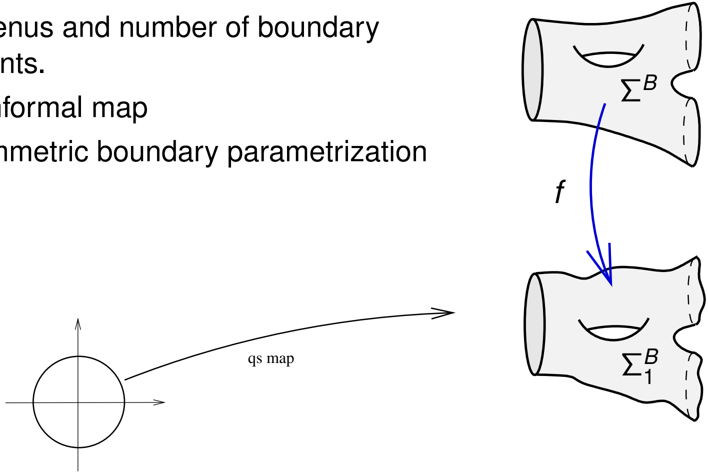
Basic Objects

- Riemann Surfaces with boundary
- Fix the genus and number of boundary components.
- Quasiconformal map



Basic Objects

- Riemann Surfaces with boundary
- Fix the genus and number of boundary components.
- Quasiconformal map
- Quasisymmetric boundary parametrization



Teichmüller Space = space of Riemann surfaces

Fix a base Riemann surface Σ.
Given Σ_1 and quasiconformal $f : \Sigma \to \Sigma_1$, write (Σ, f, Σ_1).

Teichmüller Space = space of Riemann surfaces

Fix a base Riemann surface Σ.
Given Σ_1 and quasiconformal $f : \Sigma \to \Sigma_1$, write (Σ, f, Σ_1).

Definition (Teichmüller space:)

$T(\Sigma) = \{(\Sigma, f, \Sigma_1)\}/\sim$.

$$(\Sigma, f, \Sigma_1) \sim (\Sigma, g, \Sigma_2) \iff \exists \text{ conformal } \sigma : \Sigma_1 \to \Sigma_2 \text{ such that } g^{-1} \circ \sigma \circ f \approx \text{id (rel. boundary)}$$

Teichmüller Space $=$ space of Riemann surfaces

Fix a base Riemann surface Σ.
Given Σ_1 and quasiconformal $f : \Sigma \to \Sigma_1$, write (Σ, f, Σ_1).

Definition (Teichmüller space:)

$T(\Sigma) = \{(\Sigma, f, \Sigma_1)\}/\sim$.

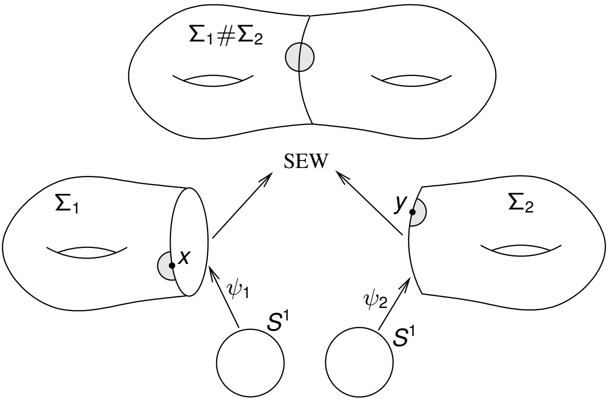
$$(\Sigma, f, \Sigma_1) \sim (\Sigma, g, \Sigma_2) \iff \exists \text{ conformal } \sigma : \Sigma_1 \to \Sigma_2 \text{ such that } g^{-1} \circ \sigma \circ f \approx \text{id (rel. boundary)}$$

Classical Facts:

1. If Σ is closed (with punctures) then $T^P(\Sigma)$ is a finite-dimensional complex manifold.
2. If Σ is a surface with boundary then $T^B(\Sigma)$ is an ∞-dimensional complex manifold.
3. Moduli space $= T(\Sigma)/$ (Mapping Class Group).

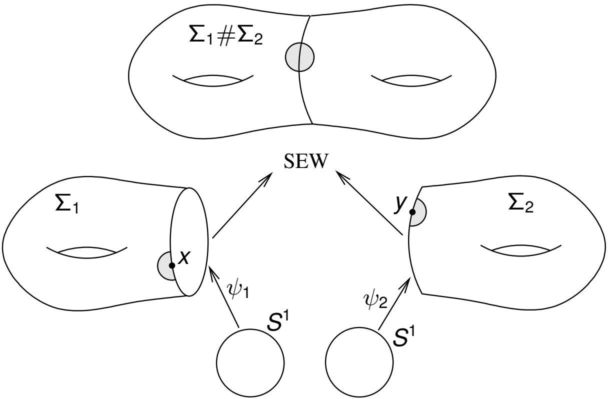
Sewing using parametrizations

$$\Sigma_1 \# \Sigma_2 = (\Sigma_1 \sqcup \Sigma_2) / \left(\psi_1^{-1}(x) = \psi_2^{-1}(y)\right)$$



Sewing using parametrizations

$$\Sigma_1 \# \Sigma_2 = (\Sigma_1 \sqcup \Sigma_2)/\left(\psi_1^{-1}(x) = \psi_2^{-1}(y)\right)$$



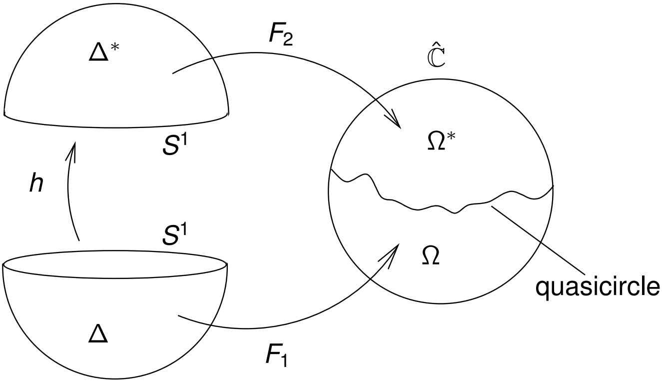
Note: If ψ_i are conformal then $\Sigma_1\#\Sigma_2$ immediately becomes a Riemann surface. This is what was previously used in CFT.

Conformal Welding – a key classical fact

Δ – unit disk, $\Delta^* = \hat{\mathbb{C}} \setminus \bar{\Delta}$, $h : S^1 \to S^1$ (quasisymmetry)

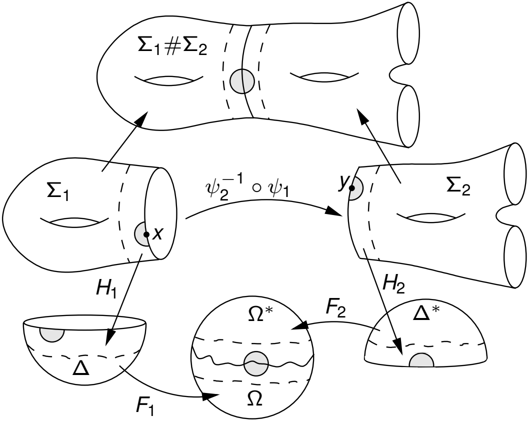
Theorem (conformal welding:)

There exists conformal maps F_1 and F_2 such that $F_2^{-1} \circ F_1 = h$ on S^1.



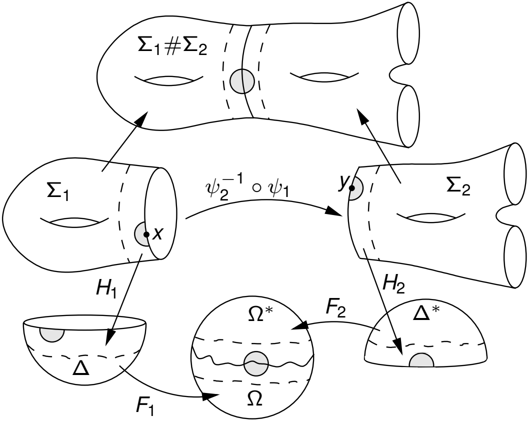
Quasisymmetric Sewing

ψ_1 and ψ_2 – quasisymmetric boundary parametrizations.
Define charts on $\Sigma_1\#\Sigma_2$ by:



Quasisymmetric Sewing

ψ_1 and ψ_2 – quasisymmetric boundary parametrizations.
Define charts on $\Sigma_1 \# \Sigma_2$ by:



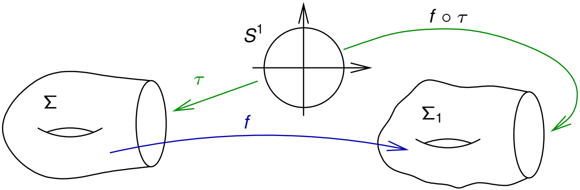
Proposition (RS 2006)

This gives the unique complex structure on $\Sigma_1 \# \Sigma_2$ which is compatible with Σ_1 and Σ_2.

Results: Teichmüller theory $\Longrightarrow$ CFT

Key idea:

Fix a QS parametrization of the base surface: $\tau : S^1 \to \partial\Sigma$.
Then $[\Sigma, f, \Sigma_1] \in T^B(\Sigma)$ **contains boundary parametrization data** for Σ_1 via $f \circ \tau$.



Theorem (R. - Schippers 2006)

(1) $T^B(\Sigma)$ / (discrete mapping class group) = rigged moduli space

(2) $T^B(\Sigma_1) \times T^B(\Sigma_2) \xrightarrow{sew} T^B(\Sigma_1 \# \Sigma_2)$ is holomorphic.

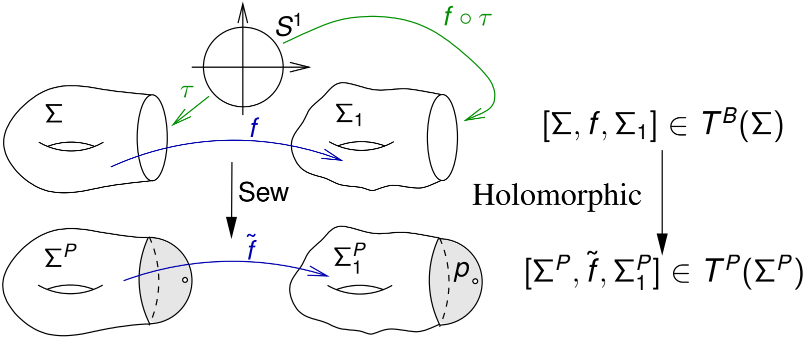
Proofs: (1) Technical. Use lambda lemma from complex dynamics etc.(2) easy

Cap Sewing: CFT $\Longrightarrow$ Teichmüller theory

Theorem (R. - Schippers 2009)

1. *T^B is a holomorphic fiber space over T^P.*
2. *The fibers are complex Banach manifolds:*

$\mathcal{O}_{qc}(\Sigma_1^P) = \{g : \mathbb{D} \to \Sigma_1^P \mid g \text{ is } 1-1, \text{ holo., has qc ext., and } f(0) = p\}$.



Relation to universal Teichmüller space

Fiber locally modeled on:

$$\mathcal{O}_{\mathrm{qc}} = \{f\colon \mathbb{D} \to \mathbb{C} \mid f \text{ is one-to-one, holomorphic, has qc extension to } \mathbb{C}, \text{ and } f(0) = 0\}.$$

Universal Teichmüller space (non-standard normalization)

$$\mathcal{D} = \{f\colon \mathbb{D} \to \overline{\mathbb{C}} \mid f \text{ is one-to-one, holomorphic, has qc extension to } \overline{\mathbb{C}}, \text{ and } f(0) = 0, f'(0) = 1, f''(0) = 0\}$$

Teichmüller curve (model due(?) to L.-P. Teo, 2004).

$$\widetilde{\mathcal{D}} = \{f\colon \mathbb{D} \to \mathbb{C} \mid f \text{ is one-to-one, holomorphic, has qc extension to } \mathbb{C}, \text{ and } f(0) = 0, f'(0) = 1\}.$$

Complex structure on $\mathcal{O}_{\mathrm{qc}}$

$$A^2_\infty(\mathbb{D}) = \{\psi(z)\colon \mathbb{D} \to \mathbb{C} \mid \psi \text{ holomorphic, } ||\psi||_{2,\infty} = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|\psi(z)| < \infty\}$$

$$A^1_\infty(\mathbb{D}) = \{\phi(z)\colon \mathbb{D} \to \mathbb{C} \mid \phi \text{ holomorphic, } ||\phi||_{1,\infty} = \sup_{z\in\mathbb{D}}(1-|z|^2)|\phi(z)| < \infty\}.$$

pre-Schwarzian: $\mathcal{A}(f) = \frac{f''}{f'}$ Schwarzian: $\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$

Embeddings: $\mathcal{S}\colon \mathcal{D} \to A^2_\infty(\mathbb{D})$ and $\mathcal{A}\colon \widetilde{\mathcal{D}} \to A^1_\infty(\mathbb{D})$

The complex structures are compatible (L.-P. Teo 2004).

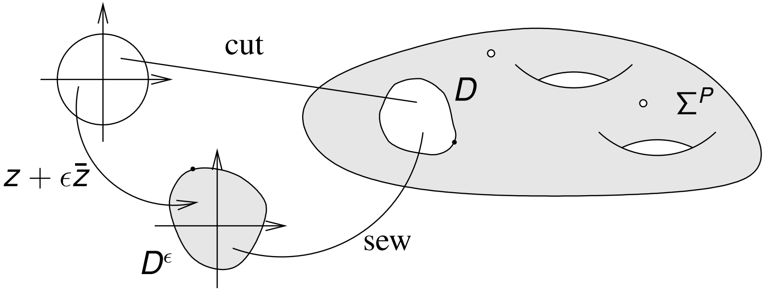
Corollary (R. - Schippers 2008)

The embedding

$$\begin{aligned} \chi\colon \mathcal{O}_{qc} &\to A^1_\infty(\mathbb{D}) \oplus \mathbb{C} \\ f &\mapsto \left(\mathcal{A}(f), f'(0)\right). \end{aligned}$$

defines a compatible complex structure on $\mathcal{O}_{qc}$

Schiffer variation = coords for Teichmüller space



$\Sigma^\epsilon = (\Sigma^P \setminus D)\#D^\epsilon$. Let $N = 3g - 3 + n$.

Theorem (Gardiner 1975, Nag 1985)

Variation on any N disks, $(\epsilon_1, \ldots \epsilon_N) \mapsto \Sigma^{(\epsilon_1, \ldots \epsilon_N)}$, gives local holomorphic coordinates for the Teichmüller space of punctured surfaces.

Fix $[\Sigma^P, f, \Sigma_1^P] \in T(\Sigma^P)$. Choose $\Omega \subset \mathbb{C}^N$ and N Schiffer variation disks on Σ_1^P. Let $U \subset \mathcal{O}_{\mathrm{qc}}(\Sigma_*)$. **Let $\Sigma_\phi^\epsilon = \Sigma_1^\epsilon \setminus \phi(\mathbb{D})$.**

Theorem (R. - Schippers 2009)

The map

$$\Omega \times U \to T^B(\Sigma)$$
$$(\epsilon, \phi) \mapsto [\Sigma, \tilde{f}, \Sigma_\phi^\epsilon]$$

is a local holomorphic coordinate chart on $T^B(\Sigma)$