Schiffer Variation in Teichmüller Space, Determinant Line Bundles and Modular Functors

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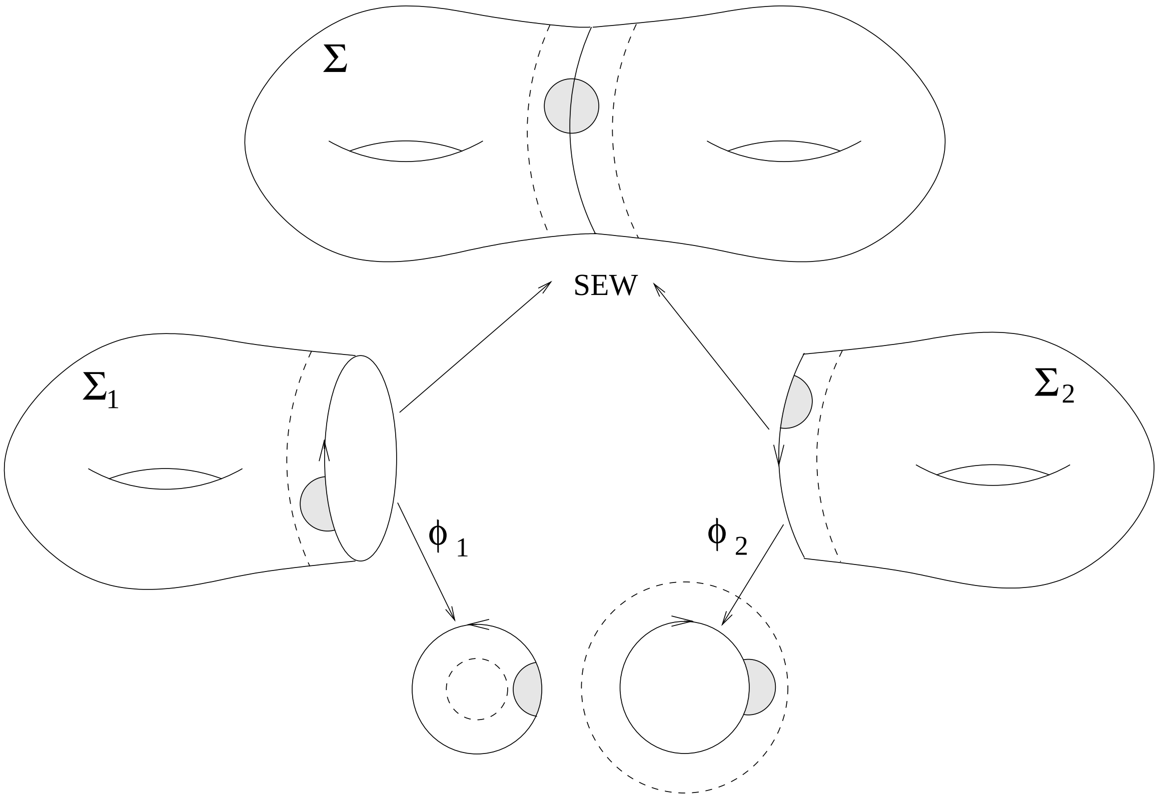
Outline

- ♢ Informal statement of the problems
- ♢ Motivation and Applications
  - – Conformal Field Theory
  - – Modular Functors
- ♢ Background geometry
  - – Quasiconformal Teichmüller Theory
  - – Schiffer Variation
- ♢ Idea of constructions and proofs
- ♢ Summary of results

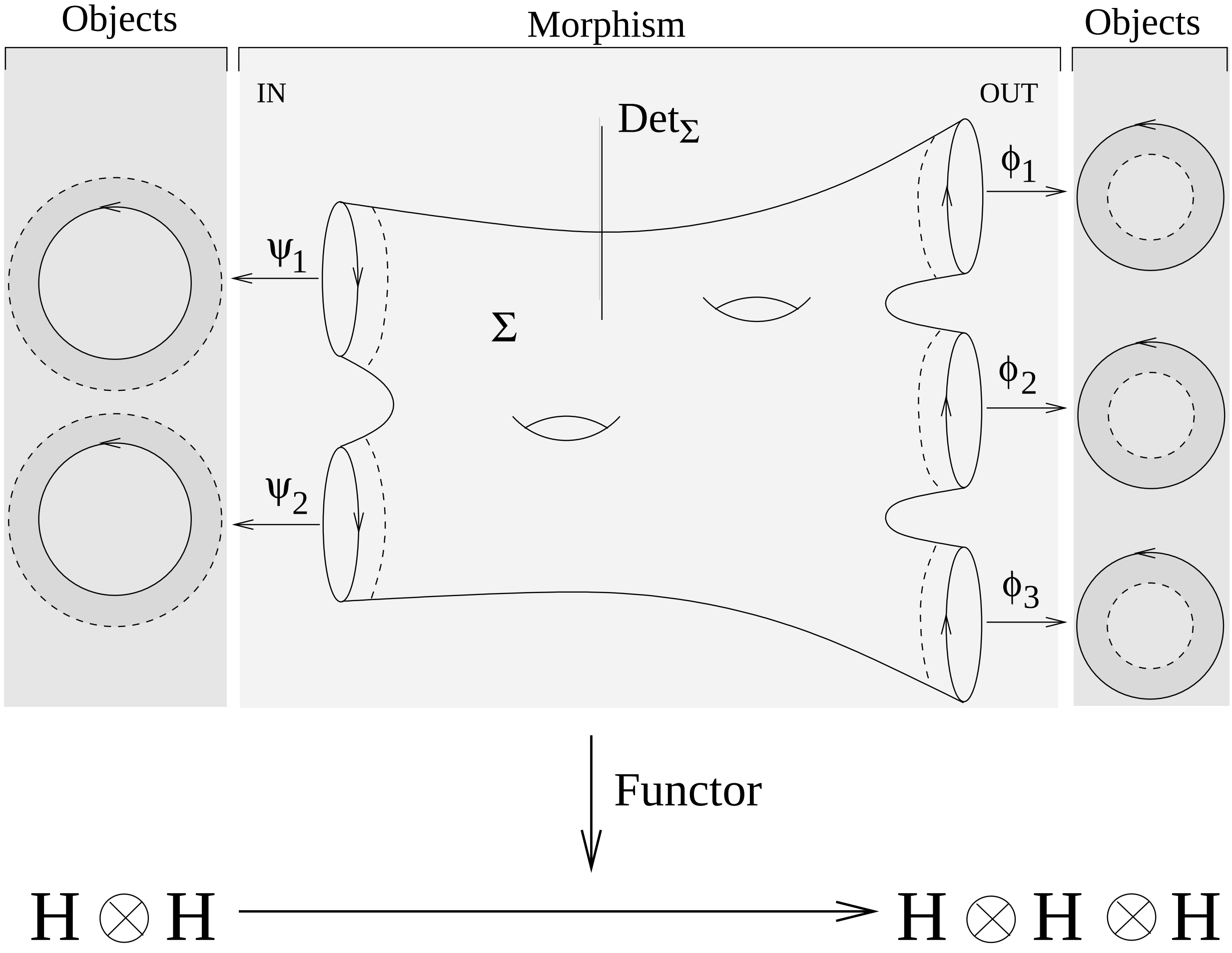
Problems

1. Moduli space of Riemann surfaces with analytically parametrized boundary.
   - Complex manifold
   - Useful/explicit complex structure
2. Determinant line bundle over the moduli space.
3. Holomorphicity of the sewing operation.



Motivation and Applications

Construction of conformal field theory (CFT).



Basic mathematical objects:

- Holomorphic (chiral) conformal field theories
- Modular functors

Genus 0 and 1 theory essentially complete using vertex operator algebras.

Background: Quasiconformal maps

Generalize conformal maps.

A homeomorphism $w : \mathbb{C} \to \mathbb{C}$ is *quasiconformal* iff

1. Geometric: The circular dilation
$$H(z) = \limsup_{r \to 0^+} \frac{\max_\xi\{|w(\xi) - w(z)| : |\xi - z| = r\}}{\min_\xi\{|w(\xi) - w(z)| : |\xi - z| = r\}}$$
is globally bounded by some $K < \infty$

OR (equivalence is a hard theorem)

2. Analytic:
$$\left|\frac{\partial w(z)}{\partial \bar{z}}\right| \leq k \left|\frac{\partial w(z)}{\partial z}\right|$$
almost everywhere for some $k \in [0, 1)$.

Easy: w smooth implies $K \geq (1 + k)/(1 - k)$.

Beltrami equation:

$$\boxed{\frac{\partial w(z)}{\partial \bar{z}} = \mu(z) \frac{\partial w(z)}{\partial z}}$$

w is quasiconformal $\Longleftrightarrow \|\mu\|_\infty < 1$.

Theorem: Given $\mu \in L^\infty(\mathbb{C})_1$ there exists a unique solution w^μ fixing 0, 1 and ∞.

Important theorem: w^μ depends analytically on μ.

Background: Teichmüller space

$M(g,n)$ = collection of Riemann surface of genus g, with n ordered punctures. Assume $2g-2+n>0$.

Moduli space: $M(g,n)$/conformal equivalence

Fact: There is a quasiconformal map between any two surfaces in $M(g,n)$.

Fix a reference surface Σ.

$$\boxed{\widehat{M}(\Sigma) = \{\ (\Sigma, f, \Sigma_1)\,|\, f: \Sigma \to \Sigma_1 \text{ is quasiconformal }\}}$$

Teichmüller equivalence: $(\Sigma, f, \Sigma_1) \sim_T (\Sigma, g, \Sigma_2) \Longleftrightarrow$

$$\begin{tikzcd}
\Sigma \arrow[r, dotted, "\simeq \text{id}"] \arrow[d, "f"'] & \Sigma \arrow[d, "g"] \\
\Sigma_1 \arrow[r, "\sigma\text{-holo}"] & \Sigma_2
\end{tikzcd}$$

where $g^{-1} \circ \sigma \circ f$ is homotopic to the identity.

Teichmüller space $\equiv T(\Sigma) = \widehat{M}(\Sigma)/\sim_T$

$$\boxed{\textbf{Theorem: } \text{If } 2g-2+n>0 \text{ then the Teichmüller space is a complex manifold of dimension } 3g-3+n.}$$

Mapping class group $\equiv \mathrm{MCG}(\Sigma) = \mathrm{Diff}^+(\Sigma)/\,\mathrm{Diff}_0^+(\Sigma)$

Fact: Moduli space $\simeq T(\Sigma)/\,\mathrm{MCG}(\Sigma)$

Background: QC maps ↔ Teichmüller space

U - upper half-plane ; G - Fuchsian group

Let $\Sigma = U/G$.   Example: torus $= U/\{z \mapsto z+1, z \mapsto z+\tau\}$

$L^\infty_{(-1,1)}(\Sigma)$ - Differential forms of type $(-1,1)$.

$L^\infty(U,G)$ - compatible with G such that

$$\begin{tikzcd} L^\infty(U,G) \arrow[r, bend left, "\text{quotient}"] \arrow[r, phantom, "1-1"] & L^\infty_{(-1,1)}(\Sigma) \arrow[l, bend left, "\text{lift}"] \end{tikzcd}$$

Choose $\mu \in L^\infty(U,G)_1$.

Let

$$G^\mu = w^\mu G (w^\mu)^{-1}.$$

Obtain a new Riemann surface

$$\boxed{\boxed{\Sigma^\mu = w^\mu(U)/G^\mu}}$$

and a quasiconformal map

$$f^\mu = w^\mu_* : \Sigma \longrightarrow \Sigma^\mu.$$

Conversely, given a quasiconformal map $f : \Sigma \to \Sigma_1$ we get

$$\mu_f = \frac{\partial f}{\partial \bar{z}} \Big/ \frac{\partial f}{\partial z} \in L^\infty_{(-1,1)}(\Sigma)_1.$$

$$\boxed{L^\infty(U,G)_1 \xleftrightarrow{1-1} \left\{ \ (\Sigma, f, \Sigma_1) \mid f\text{-quasiconformal} \ \right\}}$$

Background: Complex structure

Let $\Sigma = U/G$.

There is an equivalence relation on $L^\infty(U,G)_1$ such that

$$\begin{tikzcd}
T(G) \equiv L^\infty(U,G)_1/\sim \arrow[rr, leftrightarrow, "1-1"] & & T(\Sigma) \\
& {[\mu]} \arrow[r, mapsto] & {[\Sigma, f^\mu, \Sigma^\mu]}
\end{tikzcd}$$

With the canonical complex structure on $T(\Sigma)$, the projection

$$L(U,G)_1 \longrightarrow T(\Sigma)$$

is holomorphic.

This means that if $t \mapsto \mu_t$ is holomorphic then

$$\begin{tikzcd}
& w^{\mu_t} \arrow[d, leftrightarrow] & \\
t \arrow[r, mapsto] & \mu_t \arrow[r, mapsto] & {[\Sigma, f^{\mu_t}, \Sigma^{\mu_t}]}
\end{tikzcd}$$

gives a holomorphic map $\mathbb{C} \to T(\Sigma)$.

Upshot: Work with holomorphic families of quasiconformal maps.

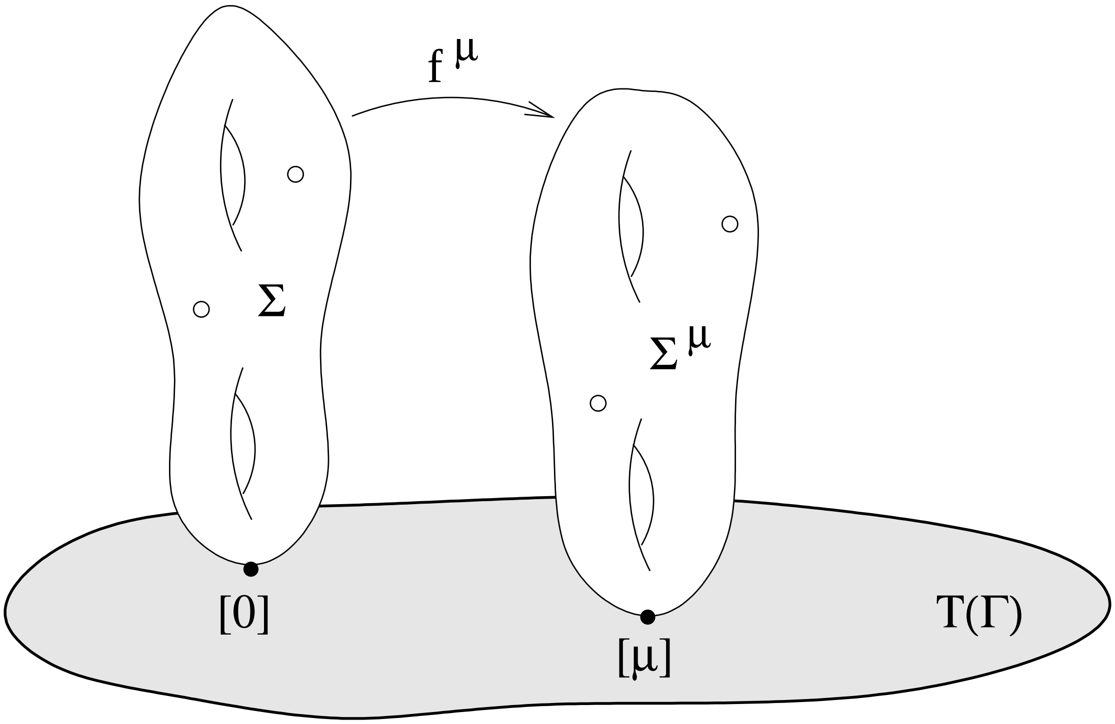
Background: Teichmüller curve

Important Fact: Σ^μ only depends on the equivalence class $[\mu]$. (higher-genus uniformization)

Theorem: The space

$$V(G) = \{([\mu], z) \mid z \in \Sigma^\mu, [\mu] \in T(G)\}$$

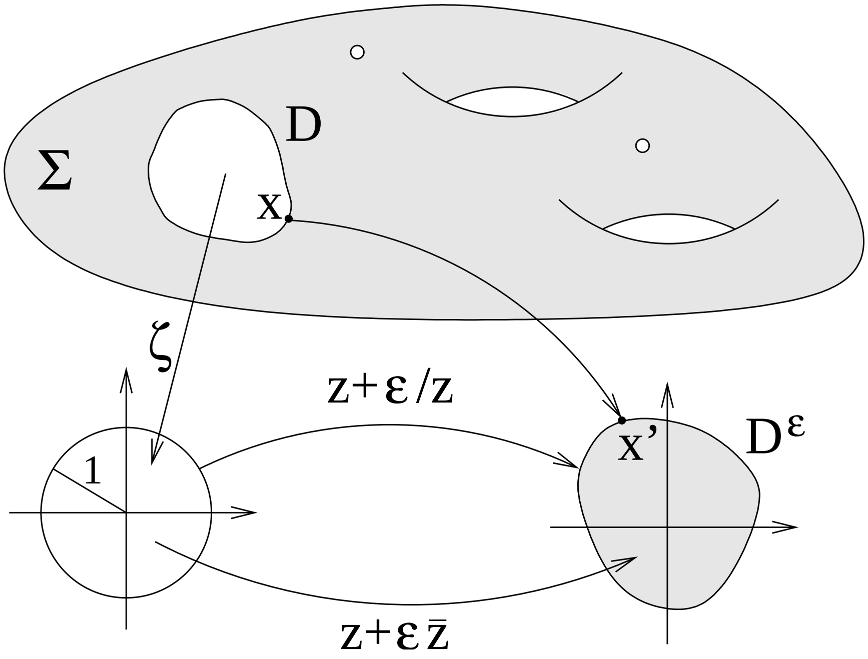
is a holomorphic fiber space over $T(G) \simeq T(\Sigma)$.



Important theorem: $V(G)$ is universal: Any marked holomorphic family of Riemann surfaces maps holomorphically into $V(G)$.

Background: Schiffer variation

Result: Get a holomorphic coordinate chart on $T(\Sigma)$ by performing by performing *local* variations on Σ.



$\epsilon \in \mathbb{C}$; $w^\epsilon = z + \epsilon\bar{z}$; $D^\epsilon = w^\epsilon$(unit ball)

For $|\epsilon|$ small, let

$$\boxed{\Sigma^\epsilon = (\Sigma \setminus D) \sqcup D^\epsilon / \sim}$$

Important theorem: For any disks $D_1, \ldots D_{3g-3+n}$, the $\epsilon_1, \ldots, \epsilon_{3g-3+n}$ give a holomorphic chart for $T(\Sigma)$.

Easy part of proof: The quasiconformal map

$$w^\epsilon(z) = z + \epsilon\bar{z}$$

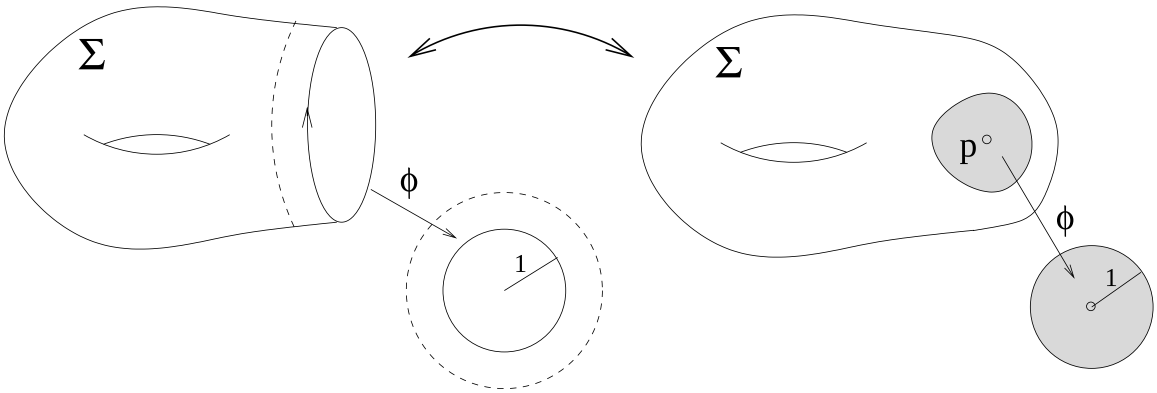
is analytic in ϵ.

Teichmüller space of rigged surfaces

Data: $(\Sigma, p_1, \ldots, p_n, \phi_1, \ldots, \phi_n)$ - Riemann surface with oriented ordered punctures and local coordinates.

$$\phi : \text{neighborhood of } p_i \longmapsto \mathbb{C}\backslash\{0\} \quad \text{or} \quad \widehat{\mathbb{C}} \setminus \{\infty\}$$

$\approx$ Equivalent to a surface with analytically parametrized boundary.



$$\widetilde{M}(\Sigma) = \big\{(\Sigma, f, \Sigma_1, \phi) \,|\, f : \Sigma \to \Sigma_1 \text{ is quasiconformal}\big\}$$

Teichmüller equivalence:

$(\Sigma, f, \Sigma_1, \phi) \sim_T (\Sigma, g, \Sigma_2, \psi) \Longleftrightarrow$

$$\begin{tikzcd}
\Sigma \arrow[r, dotted, "\simeq \text{id}"] \arrow[d, "f"'] & \Sigma \arrow[d, "g"] \\
(\Sigma_1, \phi) \arrow[r, "\sigma\text{-holo}"] & (\Sigma_2, \psi)
\end{tikzcd}$$

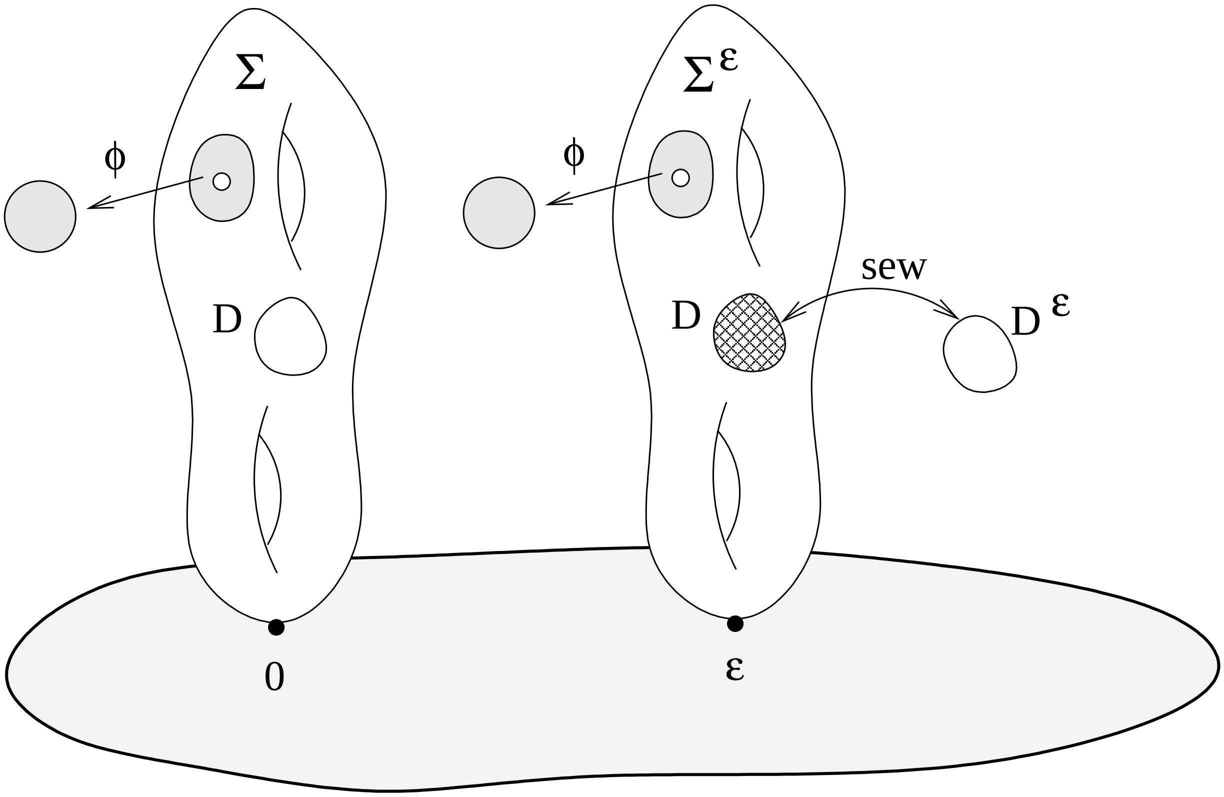
where $\phi = \psi \circ \sigma$.

Teichmüller space $\equiv \widetilde{T}(\Sigma) = \widetilde{M}(\Sigma)/\sim_T$

Difficulty: Cannot compare local coordinates as $\psi \circ f$ is not analytic.

Complex Structure on $\widetilde{T}(\Sigma)$

$\mathcal{S}$ - Schiffer family of surfaces.



Neighborhoods in $\widetilde{T}(\Sigma)$ of the form $[\Sigma, \nu^{\epsilon} \circ f, \Sigma_1^{\epsilon}, \phi_t]$.

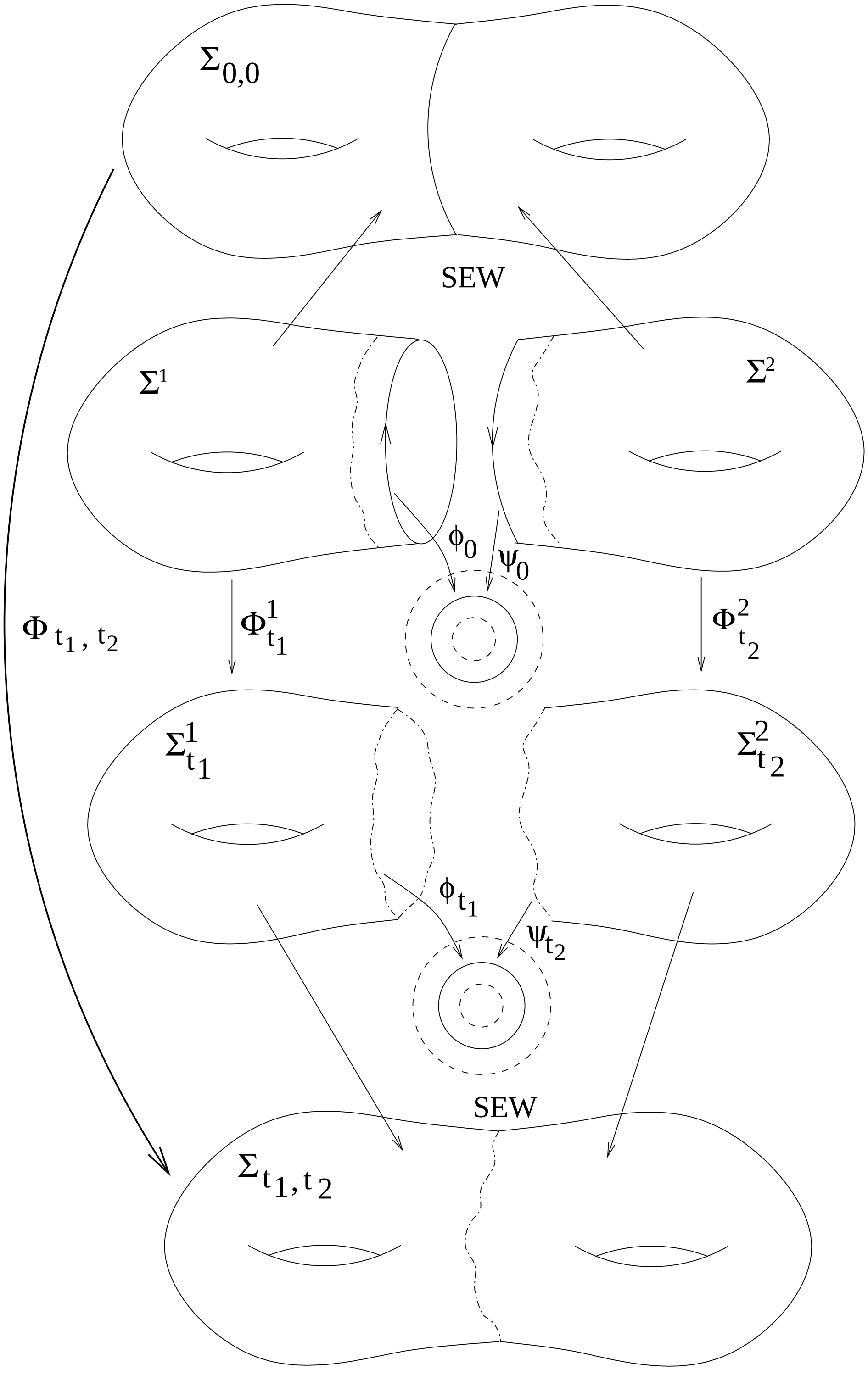
Transitions function: How to compare local coordinates on two different families that intersect?

Use the universality of $V(G)$.

$$\begin{tikzcd}
\mathcal{S}_1 \arrow[dr, "F_1"] & \\
& V(G) \\
\mathcal{S}_2 \arrow[ur, "F_2"] &
\end{tikzcd}$$

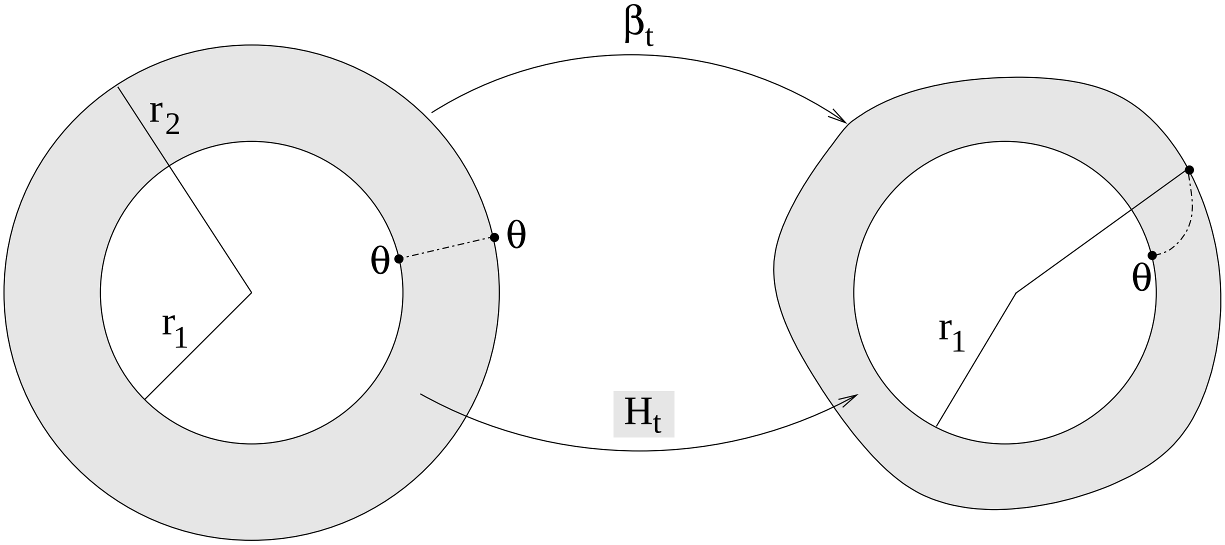
Holomorphicity of sewing

Method: Construct a quasiconformal map that depends holomorphically on the sewing parameters.



Analyticity of sewing

Idea: Reduce the problem to producing a quasiconformal map between annuli.



β_t - Family of analytic functions depending analytically on the parameter t. Also $\beta_0 =$ identity.

Problem: Construct a family of quasiconformal maps H_t that depend analytically on t and such that

$$H_0 = \text{identity}$$
$$H_t = \text{identity on the inner circle}$$
$$H_t = \beta_t \text{ on the outer circle}$$

For $|t|$ small, β_t is *close* to the identity.

H_t can be constructed explicitly for $|t|$ small.

Direct computation (or the λ−lemma) shows that H_t is quasiconformal.

Determinant line bundle

V - vector space, $dim(V) = n$. $\mathrm{Det}(V) = \bigwedge^{top}(V)$.

$\bar{\partial} \oplus \mathrm{pr} : \Omega^0(\Sigma) \to \Omega^{0,1}(\Sigma) \oplus \Omega^0_+(\partial\Sigma)$ - Fredholm operator

$$\boxed{\mathrm{Det}(\Sigma) = \mathrm{Det}(\mathrm{Ker}(\bar{\partial} \oplus \mathrm{pr}))^* \otimes \mathrm{Det}(\mathrm{Coker}(\bar{\partial} \oplus \mathrm{pr}))}$$

Genus 0 results: (Huang 1997) Determinant lines form a holomorphic bundle over the moduli space. Sewing isomorphism $\mathrm{Det}(\Sigma_1) \otimes \mathrm{Det}(\Sigma_2) \simeq \mathrm{Det}(\Sigma_1 \# \Sigma_2)$ is holomorphic.

$$\boxed{\textbf{Definition: } \mathrm{Det}([\Sigma, f, \Sigma_1, \phi]) = \mathrm{Det}(\Sigma^\mu).}$$

Use pants decomposition together with Schiffer variation to define the Bundle trivialization.

Choose a pants decomposition such that

$$\Sigma^\epsilon = \Sigma_1'^\epsilon \# \Sigma_2' \# \cdots \# \Sigma_n'$$

In term of the canonical representatives

$$\Sigma^{\mu_\epsilon} = \Sigma_1^\epsilon \# \Sigma_2 \# \cdots \# \Sigma_n$$

Trivialization:

$$\boxed{\mathrm{Det}(\Sigma^{\mu_\epsilon}) \longrightarrow \mathrm{Det}(\Sigma_1^\epsilon) \otimes \mathrm{Det}(\Sigma_2) \cdots \otimes \mathrm{Det}(\Sigma_n) \longrightarrow \mathbb{C}}$$

Determinant line bundle - Holomorphicity

Hatcher and Thurston (1980): Pants decompositions are related by two moves in genus 0 and 1.

Associativity of sewing then reduces the transitions function to genus 0 and 1.

$$\begin{tikzcd}
\mathrm{Det}(\Sigma) \arrow[rr] \arrow[drr] \arrow[ddr] \arrow[ddd] & & \mathrm{Det}(\Sigma_1)\otimes\cdots\otimes\mathrm{Det}(\Sigma_n) \arrow[r] \arrow[d, "\text{A-move}"', "\ell_A"] & \mathbb{C} \\
& & \text{step 1} \arrow[dl, bend left, "\text{S-move}"', "\ell_S"] & \\
& \text{step 2} \arrow[dl, dotted] & & \\
\mathrm{Det}(\Sigma_1')\otimes\cdots\otimes\mathrm{Det}(\Sigma_n') \arrow[d] & & & \\
\mathbb{C} & & &
\end{tikzcd}$$

Genus-zero sewing isomorphism is holomorphic.

Genus-one is done using similar methods. Non-trivial part relies on some classical results on Cauchy-type kernels and the Plemelj-Sokhotski formula on Riemann surfaces.

These methods are also needed to prove the associativity of sewing for higher genus.

$\boxed{\text{Holomorphicity of sewing - now easy.}}$

Summary of Results

$(\Sigma, p_1, \ldots, p_n, \phi_1, \ldots, \phi_n)$ - Riemann surface of genus g with punctures and local coordinates. $2g-2+n>0$

$\Sigma = U/G$

$\widetilde{T}(\Sigma)$ - rigged Teichmüller space.

$[\mu] \in T(G) = L^\infty(U, G)/\sim$

$\Sigma^\mu = w^\mu(U)/w^\mu G (w^\mu)^{-1}$

Theorem: The Teichmüller space and moduli space of Riemann surfaces with analytically parametrized boundaries is an infinite-dimensional complex manifold.

Theorem: The sewing operation is a holomorphic map

$$\widetilde{T}(\Sigma_1) \times \widetilde{T}(\Sigma_2) \longrightarrow \widetilde{T}(\Sigma_1 \# \Sigma_2)$$

Theorem: The determinant lines defined by

$$\mathrm{Det}_{[\Sigma, f, \Sigma_1, \phi]} = \mathrm{Det}(\Sigma^\mu)$$

form a holomorphic line bundle over $\widetilde{T}(\Sigma)$. Moreover the sewing operation is holomorphic on this bundle.

Corollary: The mapping class group acts without fixed points. So above results pass to the moduli space.

Application: Techniques can be used to study holomorphic modular functors. Construction of a holomorphic (projectively) flat connection.